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THREE THEOREMS ON THE ERROR OF SOLUTION OF DIFFERENT EQUATIONS OF THE THEORY OF SHELLS WITH A SINGULAR RIGHT SIDE

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It is known that the behavior of shells under the action of loads of a singular character (in particular, concentrated and piecewise-constant) can be described satisfactorily by differential equations in eighth-order or higher-order partial derivatives, depending on the rigorousness of the initial hypotheses. Relaxation of the hypotheses made the equations of the theory of shells similar to the equations of the three-dimensional theory of elasticity. The theory of shells was constructed as early as the last century, when effective calculation facilities were practically lacking and the necessity of refining models was increasing. The high order of the equations of the theory of shells generated a need for the development of new theories, which reduced, as a rule, to simplification of the equations of the general theory of shells. This is true, for example, for the theory of shallow shells (the Vlasov–Donnel equations for cylindrical shells) and for the semi-momentless theory of shells [1]. These equations are used independently in solution of problems of strength and stability of structures, and also as components in construction of solutions by asymptotic synthesis methods (ASM) [2, 3], in which they play the role of the so-called elementary stress states [4]. The latter are "glued" by a certain procedure [2] to give, as a result, the total stress-strain state of the shell. This approach proved to be very effective in determining the stress-strain state of shells subjected to the action of loads of a singular character. In the present paper, the upper bounds of the error of solutions found using the ASM are obtained.

We consider the asymptotic error of approximate equations. According to [4], by asymptotic error is meant the modulus of the ratio of the largest omitted term of the governing equation to the largest retained term. Let there be a differential equation in partial derivatives of the rth order (r is an even number) with a small parameter h_*^2 at the higher derivative or ahead of the operator containing the higher derivatives:

$$(h_*^2\mathcal{L} + M)\Phi(\alpha,\beta) = af(\alpha,\beta). \tag{1}$$

Here $\mathcal{L}$ is a differential operator that contains partial derivatives of order r and lower, M is an operator that contains derivatives of order $(1/2)r$, $f(\alpha,\beta)$ is a piecewise-constant function, and a is a constant coefficient.

In the case of circular cylindrical shells, the operators $\mathcal{L}$ and M take the following form:
in the general theory of shells

$$\mathcal{L} = \nabla^2\nabla^2(\nabla^2+1)^2 - 2(1-\nu)\Big(\frac{\partial^4}{\partial\alpha^4} - \frac{\partial^4}{\partial\alpha^2\partial\beta^2}\Big)\nabla^2, \quad M = \frac{\partial^4}{\partial\alpha^4}; \tag{2}$$

in the theory of shallow shells (the Vlasov–Donnel equations)

$$\mathcal{L} = \nabla^2\nabla^2\nabla^2\nabla^2, \quad M = \frac{\partial^4}{\partial\alpha^4}, \quad \nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}; \tag{3}$$

in the semimomentless theory of shells

$$\mathcal{L} = \frac{\partial^4}{\partial\beta^4}\Big(\frac{\partial^2}{\partial\beta^2}+1\Big)^2, \quad M = \frac{\partial^4}{\partial\alpha^4}; \tag{4}$$

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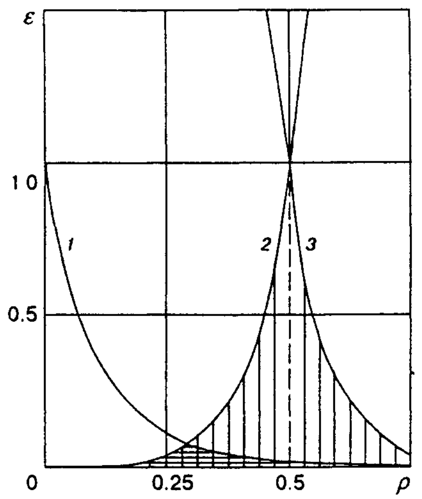

Fig. 1

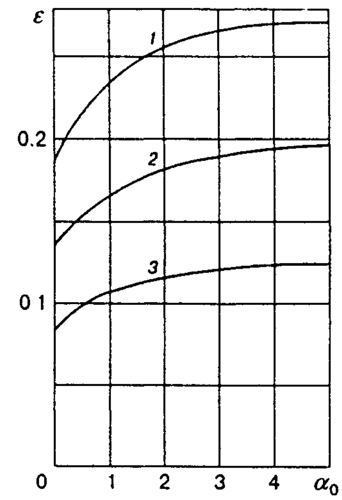

Fig. 2

in the theory of simple edge effect

$$\mathcal{L} = \frac{\partial^4}{\partial \alpha^4}, \quad M = 1; \tag{5}$$

and for a flexural or tangential state

$$\mathcal{L} = \nabla^2 \nabla^2, \quad M = 0. \tag{6}$$

Let the stress-strain state occurring in a shell be characterized by a quantity that will be called the variability factor of the stress state. The variability in a given region of the shell surface means the modulus of the ratio of the average value of the derivative of a function to the average value of the function. Let the variability factor ρ be related to the relative shell thickness and the harmonic number n in the Fourier series expansion of the function $f(\alpha, \beta)$ in terms of the circumferential coordinate β by the relation [4]:

$$n = h_*^{-\rho}, \quad h_*^2 = h^2/12\,R^2. \tag{7}$$

In relation (7), the quantity n characterizes the variability of the nth harmonics of the Fourier expansion of the function $f(\alpha, \beta)$ and also the variability of solutions in the form of series for the resolvent function and its derivatives.

Depending on the quantity ρ, one can obtain different approximate equations of the theory of shells, whose operators are written in (2)–(6). These equations, as was noted, are of independent interest and describe one or another elementary stress in one of the ASM.

The behavior of the asymptotic error ε versus ρ is shown in Fig. 1 (curve 1 refers to the shallow shell theory, curve 2 refers to semimomentless theory and edge effect, and curve 3 to equations of the flexural and tangential states) [2, 3]. The equality of the asymptotic error to unity for $\rho = 1/2$ means that in going over from full equations to approximate equations, we omit terms of the same order compared with the remaining terms in the characteristic equation (and in the governing equation, accordingly).

Suppose we constructed a stress-strain state of a shell under the action of a concentrated and piecewise-constant load on it, using as a basis equations of the general theory of shells, and then equations of shallow shells (the Vlasov–Donnel equations). Using the Fourier integral method, we write a solution for the case of a piecewise-constant function $f(\alpha, \beta)$. Then, using equations of the general theory of shells, we obtain

$$\Phi(\alpha, \beta) = \frac{6(1-\nu^2)}{\pi \alpha_0 \beta_0} \left(\frac{R}{h}\right)^3 \frac{P}{ER} \sum_{n=0}^{\infty} \theta_n \cos kn\beta \int_0^{\infty} \frac{\sin \alpha_0 \lambda}{\lambda \mathcal{L}(\lambda, n)} \cos \alpha\lambda \, d\lambda,$$

$$ERP^{-1}w(\alpha,\beta) = \frac{6(1-\nu^2)}{\pi\alpha_0\beta_0}\left(\frac{R}{h}\right)^3 \sum_{n=0}^{\infty} \theta_n \cos kn\beta \int_0^{\infty} \frac{w(\lambda,n)\sin\alpha_0\lambda}{\lambda\mathcal{L}(\lambda,n)} \cos\alpha\lambda d\lambda, \tag{8}$$

$$P^{-1}G_i(\alpha,\beta) = \frac{1}{2\pi\alpha_0\beta_0} \sum_{n=0}^{\infty} \theta_n \cos kn\beta \int_0^{\infty} \frac{g_i(\lambda,n)\sin\alpha_0\lambda}{\lambda\mathcal{L}(\lambda,n)} \cos\alpha\lambda d\lambda.$$

Here

$$\mathcal{L}(\lambda,n) = (\lambda^2+k^2n^2-1)^2(\lambda^2+k^2n^2)^2 + 2(1-\nu)\lambda^2(\lambda^4-k^4n^4) + (1-\nu^2)c^{-2}\lambda^4,$$
$$w(\lambda,n) = (\lambda^2+k^2n^2)^2, \quad g_1(\lambda,n) = (\lambda^2+\nu k^2n^2-\nu)w(\lambda,n) + (1-\nu^2)\lambda^2k^2n^2,$$
$$g_2(\lambda,n) = (k^2n^2+\nu\lambda^2-1)w(\lambda,n), \quad c^2 = h_*^2, \tag{9}$$
$$\theta_n = \frac{k\beta_0}{\pi} \quad (n=0), \qquad \theta_n = \frac{2}{\pi n}\sin kn\beta_0 \quad (n=1,\,2,\,3,\,\ldots);$$

P is the total load acting on one of the k rectangular regions located in the initial section of an infinitely long shell; and $2\alpha_0 R$ and $2\beta_0 R$ is the length of the loaded region of the shell surface in the longitudinal and axial directions, respectively.

We can write a solution for the case of k concentrated radial forces P according to (8) if we pass to the limit $\alpha_0 \to 0$ and $\beta_0 \to 0$.

The actual error in calculating the resolvent function and the desired factors in the shell using approximate equations is defined as $\varepsilon = 1-\eta$, where η is the ratio of the value of some factor (displacement, force or moment) determined from these equations to the corresponding value obtained from the theory of shells. Then the Vlasov–Donnel equations lead to

Theorem 1. *The error ε of the resolving function $\Phi(\alpha,\beta)$ and its derivatives to the $(r-1)$th order inclusive is minimal for concentrated action and increases with an increase in the loaded region, remaining the least for higher-order derivatives and for the desired factors containing these derivatives.*

Hence it follows that in formulas (8) the greatest error appears for the normal displacement w and the least for the longitudinal G_1 and circumferential G_2 bending moments. This is supported by numerous results for finite length shells [5].

We consider loading of a shell along segments of the generatrix of different length. The load is symmetric about the middle of the shell which is hinge-supported at the edges.

Figure 2 shows the error of the radial displacement ε for shells with different thickness versus the length of the loaded generatrix segment. Here curves 1–3 correspond to $R/h = 15$, 100, and 300, $l/R = 10$, where l and R are the length and radius of the shell.

For the longitudinal force, the longitudinal moment, and the circumferential moment (curves 1–3, respectively) similar dependences are given in Fig. 3 for a shell with $l/R = 10$ and $R/h = 15$. From Figs. 2 and 3 one can see that the greatest error is obtained for the radial displacement. For all factors, the errors increase with an increase in l/R and in the length of the loaded segment, reaching the largest value for loading over the entire generatrix of the shell. For thinner shells, the error ε is smaller, and this agrees with the conclusions of [6]. When shells are loaded over square regions ($\alpha_0 = \beta_0 = \delta_0$), as in the case of loading along the generatrix segments, the greatest error ε is obtained for the radial displacement.

Figure 4 shows the error of the radial displacement for loaded regions of various dimensions versus the relative thickness of a shell with $l/R = 8$; the load is applied at the middle of the shell; curves 1–5 correspond to $\delta_0 = 0.25$, 0.125, 0.0625, 0.03125, and 0 ($\delta_0 = 0$ corresponds to the action of a concentrated force). Note one important circumstance responsible for the choice of an infinitely long shell as the subject of study in this work: for a fixed shell thickness and fixed dimensions of the loaded region, an increase in the shell length always leads to an increase in the error.

Note that the solution based on the Vlasov–Donnel equations is obtained by substitution of simpler equations for relations (9):

$$\mathcal{L}(\lambda,n) = (\lambda^2+k^2n^2)^4 + (1-\nu^2)c^{-2}\lambda^4, \quad g_1(\lambda,n) = (\lambda^2+\nu k^2n^2)w(\lambda,n), \quad g_2(\lambda,n) = (k^2n^2+\nu\lambda^2)w(\lambda,n). \tag{10}$$

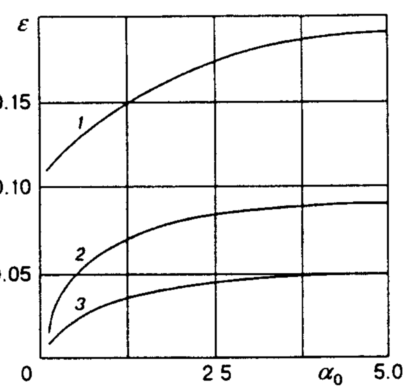

Fig. 3

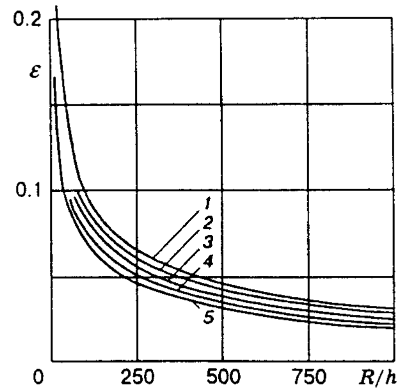

Fig. 4

It is of interest to compare the errors ε of the resolvent function, its derivatives, and the desired force and deformation factors for rectangular ($|\alpha| \leqslant \alpha_0$, $|\beta| \leqslant \beta_0$) and square ($|\alpha| \leqslant \delta_0$, $|\beta| \leqslant \delta_0$) loaded regions. where $\delta_0 = \max(\alpha_0, \beta_0)$). In this case, the following theorem holds

Theorem 2. *The error ε in loading a shell over a square region is the upper bound of the error ε for any rectangular region inscribed in this square region.*

Let now a stress-strain state be constructed on the basis of one of the two ASM rather than on the differential equation of the general theory of shells.

When the first of the ASM is used, the full stress state for $n \leqslant \bar{n}$ is obtained on the basis of equations of semimomentless theory (4) and edge-effect theory (5), and for $n \geqslant \bar{n}+1$ it is obtained on the basis of the Vlasov–Donnel equations (3). Then, instead of the solution by the general theory of shell, written in the form of (8) we obtain

$$ERP^{-1}w(\alpha,\beta) = \frac{6(1-\nu^2)}{\pi\alpha_0\beta_0}\left(\frac{R}{h}\right)^3\Bigg[\sum_{n=0}^{\bar{n}}\theta_n\cos kn\beta\int\limits_0^\infty \frac{\sin\alpha_0\lambda}{\lambda(\lambda^4+4æ^4)}\cos\alpha\lambda d\lambda + \frac{1}{4æ^4}\sum_{n=1}^{\bar{n}}\theta_n\cos kn\beta$$
$$\times\int\limits_0^\infty \frac{w_n^0\sin\alpha_0\lambda}{\lambda(\lambda^4+4\mu_n^4)}\cos\alpha\lambda d\lambda + \sum_{\bar{n}+1}^{\infty}\theta_n\cos kn\beta\int\limits_0^\infty \frac{w(\lambda,n)\sin\alpha_0\lambda}{\lambda[(\lambda^2+k^2n^2)^4+4æ^4\lambda^4]}\cos\alpha\lambda d\lambda\Bigg], \tag{11}$$
$$P^{-1}G_2(\alpha,\beta) = \frac{1}{2\pi\alpha_0\beta_0}\Bigg[\nu\sum_{n=0}^{\bar{n}}\theta_n\cos kn\beta\int\limits_0^\infty \frac{\lambda\sin\alpha_0\lambda}{\lambda^4+4æ^4}\cos\alpha\lambda d\lambda + \frac{1}{4æ^4}\sum_{n=1}^{\bar{n}}\theta_n\cos kn\beta$$
$$\times\int\limits_0^\infty \frac{g_{2n}^0\sin\alpha_0\lambda}{\lambda(\lambda^4+4\mu_n^4)}\cos\alpha\lambda d\lambda + \sum_{\bar{n}+1}^{\infty}\theta_n\cos kn\beta\int\limits_0^\infty \frac{g_2(\lambda,n)\sin\alpha_0\lambda}{\lambda[(\lambda^2+k^2n^2)^4+4æ^4\lambda^4]}\cos\alpha\lambda d\lambda\Bigg].$$

As the harmonic number $\bar{n}$ for which the solutions are "glued" according to Eqs. (3)–(5), we use the value of n found from the formula of [3] and rounded to the nearest integer:

$$n^4 = (2/k^4)\sqrt{3}(R/h). \tag{12}$$

Using the second ASM, we obtain a solution by simplifications of solution (11), more precisely, by simplification of the latter terms, which are solutions of Eq. (3). The full stress state is thus formed by the semimomentless solution, the edge effect ($n \leqslant \bar{n}$), the solution based on the Vlasov–Donnel equations ($\bar{n}+1 < n \leqslant n^*$), and the flexural state ($n \geqslant n^*+1$):

$$ERP^{-1}w(\alpha,\beta) = \frac{6(1-\nu^2)}{\pi\alpha_0\beta_0}\left(\frac{R}{h}\right)^3\Bigg[\sum_{n=0}^{\bar{n}}\theta_n\cos kn\beta\int\limits_0^\infty \frac{\sin\alpha_0\lambda}{\lambda(\lambda^4+4æ^4)}\cos\alpha\lambda d\lambda$$

$$+\frac{1}{4æ^4}\sum_{n=1}^{n}\theta_n\cos kn\beta\int_0^\infty\frac{w_n^\circ\sin\alpha_0\lambda}{\lambda(\lambda^4+4\mu_n^4)}\cos\alpha\lambda d\lambda$$

$$+\sum_{\bar{n}+1}^{n^*}\theta_n\cos kn\beta\int_0^\infty\frac{w(\lambda,n)\sin\alpha_0\lambda}{\lambda[(\lambda^2+k^2n^2)^4+4æ^4\lambda^4]}\cos\alpha\lambda d\lambda+\sum_{n^*+1}^{\infty}\theta_n\cos kn\beta\int_0^\infty\frac{\sin\alpha_0\lambda}{\lambda(\lambda^2+k^2n^2)^2}\cos\alpha\lambda d\lambda\Big],$$

$$P^{-1}G_2(\alpha,\beta)=\frac{1}{2\pi\alpha_0\beta_0}\Big[\nu\sum_{n=0}^{\bar{n}}\theta_n\cos kn\beta\int_0^\infty\frac{\lambda\sin\alpha_0\lambda}{\lambda^4+4æ^4}\cos\alpha\lambda d\lambda \tag{13}$$

$$+\frac{1}{4æ^4}\sum_{n=1}^{\bar{n}}\theta_n\cos kn\beta\int_0^\infty\frac{g_{2n}^\circ\sin\alpha_0\lambda}{\lambda(\lambda^4+4\mu_n^4)}\cos\alpha\lambda d\lambda$$

$$+\sum_{\bar{n}+1}^{n^*}\theta_n\cos kn\beta\int_0^\infty\frac{g_2(\lambda,n)\sin\alpha_0\lambda}{\lambda[(\lambda^2+k^2n^2)^4+4æ^4\lambda^4]}\cos\alpha\lambda d\lambda+\sum_{n^*+1}^{\infty}\theta_n\cos kn\beta\int_0^\infty\frac{(k^2n^2+\nu\lambda^2)\sin\alpha_0\lambda}{(\lambda^2+k^2n^2)^2}\cos\alpha\lambda d\lambda\Big].$$

In relations (11) and (13), the notation is as follows:

$$w_n^\circ=k^4n^4;\quad g_{2n}^\circ=k^4n^4(k^2n^2-1);\quad æ^4=3(1-\nu^2)(R/h)^2.$$

As the harmonic number n^*, we use the value n found from the formula of [3] and rounded to the nearest integer n:

$$n^4=(2/k^4)(1-\nu^2)(R/h)^{5/2}. \tag{14}$$

Thus, using the second ASM, one can obtain the stress-strain state of a shell by formulas (13). In this case, the solutions of the approximate equations (3)–(6) are "glued" for values $\bar{n}$, and n^* determined form formulas (12) and (14). Both methods give solutions that practically coincide with the exact solution with a much shorter time of numerical realization. In addition, the ASM make it possible to obtain convenient, readily visible analytic solutions or finite calculation formulas.

As is known, the error of the theory of shells constructed with accuracy to Kirchhoff–Love hypotheses is a quantity of order h/R as compared with unity. The question of estimation of the errors of various approximate equations and ASM arises [6–9]. Let we have a differential equation (1) with an error of order h/R compared with unity [6–8], containing a small parameter h_*^2 at the higher-order derivatives. For split equations that follow from (1), curves of variation in the asymptotic error are given in Fig. 1. The criterional value $\bar{n}$ is found from the condition of minimum of the asymptotic error, and the value n^* is found from the condition of possible neglect of terms of order $(h/R)^{1/2}$ compared with unity in Eq. (1) (the latter circumstance makes the elliptic equation of the general theory a polyharmonic equation $\nabla^8\Phi=0$). Then, the following theorem holds.

Theorem 3. *If the error of the exact equation of the theory of shells* (1) *containing the small parameter* h_*^2 *at the higher-order derivatives is of the order of* h/R *compared with unity, the error of the first and second* ASM *does not exceed a quantity of order* $(h/R)^{1/2}$ *compared with unity.*

The ample numerical material given in [2, 3, 5] and also in other sources confirms the validity of the statements formulated in the above theorems. However, the question of analytic proof of the statements formulated as the theorems remains heuristic.

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